ON POTENTIAL AND WAVE FUNCTIONS IN n DIMENSIONS

by BALTH. VAN DER POL

Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken Eindhoven/Holland

Résumé

On sait qu'il est possible d'obtenir une solution de l'équation des ondes dans l'espace à n dimensions, à partir d'une solution dans l'espace à $n + 1$ dimensions (méthode de descente, Hadamard). L'auteur donne quelques exemples nouveaux d'application de cette méthode, contenant des discontinuités, obtenus au moyen du calcul symbolique. Il montre ensuite qu'on peut également obtenir une telle solution à partir d'une solution dans l'espace à $n - 1$ dimensions („méthode de montée"). L'auteur obtient ainsi d'une manière simple et sous une forme un peu plus générale les expressions classiques du potentiel et des ondes données par Whittaker pour le cas de l'espace à trois dimensions. Les exemples donnés utilisent des fonctions de Bessel, de Hankel et des fonctions sphériques des deux types.

1. *Introduction.* It is obvious that the plane wave

$$u = e^{i\omega\left(t-\frac{x_1}{c}\right)} \tag{1}$$

can exist in any number of dimensions because it is a solution of the n-dimensional wave equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots . + \frac{\partial^2}{\partial x_n^2} - \frac{\partial^2}{c^2 \partial t}\right) u = 0, \tag{2}$$

which depends upon x_1 and t only. It is also evident that, if

$$u = u_n\,(x_1\,, x_2 \ldots . x_n) \text{ satisfies (2),}$$
$$u = u_n\,(x_1 + \xi_1, x_2 \ldots . x_n) \tag{3}$$

does also, because in (2) x_1 occurs in a differential operator only. Hence

$$u_{n-1} = \int_{\xi_1=-\infty}^{\xi_1=+\infty} u_n\,(x_1 + \xi_1\,, x_2 \ldots . x_n)\, d\xi_1$$
$$= \int_{\xi=-\infty}^{\xi=+\infty} u_n\,(\xi\,, x_2 \ldots . x_n)\, d\xi \tag{4}$$

satisfies the wave equation in $n - 1$ dimensions provided the integral converges. Thus starting with a solution u_n, one can obtain solutions in $(n-1)$, $(n-2)$.... etc. dimensions. This simple and old method is called by Hadamard[1]) *„la méthode de descente"*.

The following instances of this method (which can very simply be derived with the symbolic calculus) may be of interest.

a) If initially the wave field in three dimensions is everywhere zero, and if at the time $t = 0$ we suddenly cause the wave potential u of the whole *z-axis* to jump to the value unity, the resulting two-dimensional wave is

$$u_2 = \begin{cases} 0 & t < \frac{\rho}{c}, \\ \log\left(\frac{tc}{\rho} + \sqrt{\frac{t^2c^2}{\rho^2} - 1}\right), & t > \frac{\rho}{c}, \end{cases}$$

with $\rho^2 = x^2 + y^2$.

b) If again the initial field is originally zero everywhere, but if we suddenly cause the wave potential of the whole $z = 0$ *plane* to jump to unity, the resulting one dimensional solution becomes

$$u_1 = \begin{cases} 0 & t < \frac{|z|}{c}, \\ 2\pi\,(ct - |z|), & t > \frac{|z|}{c}. \end{cases}$$

c) Again, with initial field everywhere zero and the wave potential on the *z-axis* performing an impulsive function $\delta(t)$, i.e. always being zero except at the time $t = 0$ where it becomes infinite in such a way that

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1,$$

the resulting two dimensional wave potential becomes

$$u_2 = \begin{cases} 0, & t < \frac{\rho}{c}, \\ \frac{1}{\sqrt{t^2 - \rho^2/c^2}}, & t > \frac{\rho}{c}. \end{cases}$$

d) Finally if the same impulsive function forms the „Belegung" of the whole $z = 0$ *plane*, the resulting one dimensional solution is

$$u_1 = \begin{cases} 0, & t < \dfrac{|z|}{c}, \\ 2\pi c, & t > \dfrac{|z|}{c}. \end{cases}$$

A constant potential in one dimension can thus be considered to be caused by the radiation due to an impulse all over a plane at right angles and which occurred previously. Incidentally the cases (a) and (c) clearly demonstrate the development of a „tail" which is characteristic for two dimensional wave functions.

2. *"Méthode de montée."* Returning to the "méthode de descente" it follows from (4) that this method essentially consists of superposing wave (or potential) solutions in the direction of one axis and it is evident that the result will be independent of that coordinate and thus represents a solution in one dimension less.

We now will show that, analogous to the method described, we can apply a *"méthode de montée"* enabling one to derive wave or potential functions in $n + 1$ dimensions starting from a solution in n dimensions.

To this end we remark that, given the linearity of the equation, we may superpose solutions with any arbitrary „weights" partially or all along any *angle* and thus a new solution in $n + 1$ dimensions is obtained from a given solution in n dimensions. Regarding the time as a new coordinate, one may thus construct wave functions in n dimensions from a given potential function in the same number of dimensions. The validity of this *„méthode de montée"* is otherwise obvious from the fact that e.g. the three dimensional potential or wave equation written in cylindrical or spherical coordinates

$$y = \rho \cos \varphi, \qquad x = \rho \sin \varphi, \qquad z = z;$$
$$y = r \sin \theta \cos \varphi, \qquad x = r \sin \theta . \sin \varphi, \qquad z = r \cos \theta;$$

contains φ in a differential operator only. Hence when the two dimensional solution

$$u_2 = u\,(z, x) \tag{5}$$

be given, which we can write as

$$u_2 = u\,(z, \rho \cos \varphi),$$

a solution in three dimensions will be

$$u_3 = u\,\{z, \rho \cos (\varphi - \varphi')\}, \tag{6}$$

and finally a more general three dimensional solution is

$$u_3 = \int\limits_{\varphi'=\varphi_1}^{\varphi'=\varphi_2} u\,(z, \rho \cos(\varphi - \varphi')) . \;\psi(\varphi') . d\varphi', \tag{7}$$

where $\psi(\varphi')$ is any arbitrary „weight" function.

3. *Derivation and extension of Professor* E. T. Whittaker's *solution of the potential and wave equation.* Consider the potential equation in three dimensions. Certainly any complex function

$$u_2 = f\,(z + ix)$$

will solve it. Write this in cylindrical coordinates as

$$u_2 = f\,(z + i\rho \cos \varphi).$$

Hence a solution will be

$$u_3 = f\,(z + i\rho \cos(\varphi - \varphi'))$$

and also

$$u_3 = \int\limits_{\varphi_1}^{\varphi_2} f\,(z + i\rho \cos(\varphi - \varphi')) . \;\psi(\varphi') . d\varphi', \tag{8}$$

where $\psi(\varphi')$ is any arbitrary „weight" function and φ_1 and φ_2 are arbitrary angles. Writing (8) in rectangular coordinates we obtain:

$$u_3 = \int\limits_{\varphi_1}^{\varphi_2} f\,(z + iy \cos \varphi' + ix \sin \varphi') . \;\psi(\varphi')\, d\varphi'. \tag{9}$$

Equation (9), but with the arbitrary integration limits replaced by $\varphi_1 = 0$ and $\varphi_2 = 2\pi$, constitutes Professor Whittaker's solution of the potential equation [2]), which the discoverer applied so successfully in his „Modern Analysis" to the investigation of the equations of theoretical physics. As a simple example take

$$f\,(z + iy) = (z + iy)^{-(n+1)}, \qquad \psi = 1.$$

These functions yield immediately the potential function

$$\frac{1}{r^{n+1}} . \underset{n}{P}(\cos \theta) = \frac{1}{2\pi} \int\limits_0^{2\pi} (z + iy)^{-(n+1)}\, d\varphi'$$

$$= \frac{1}{2\pi} \int\limits_0^{2\pi} (z + i\rho \cos \varphi')^{-(n+1)} d\varphi'.$$

The arbitrary limits in (8) or (9) may, however, be taken complex so that (8) or (9) also includes e.g. the potential function

$$\frac{1}{r^{n+1}} \cdot \underset{n}{Q} (\cos \theta) = \tfrac{1}{2}i \int_{-i\infty}^{+i\infty} (z + iy)^{-(n+1)}\, d\varphi'$$

$$= \tfrac{1}{2}i \int_{-i\infty}^{+i\infty} (z + i\rho \cos \varphi')^{-(n+1)} .\, d\varphi' .$$

It is further easy to extend (9) to more dimensions. For all we have to do is to write any variable, say x_n, as

$$x_n = \rho \cos \varphi_{n,\, n+1} ,$$

where

$$\rho^2 = x_n^2 + x_{n+1}^2 ,$$

$$x_{n+1} = \rho \sin \varphi_{n,\, n+1} ,$$

thus introducing the extra dimension x_{n+1}. Hence the direct extension of (9) to four dimensions yields immediately as a general four-dimensional potential in (x, y, z, v):

$$u_4 = \int_{\varphi'=\varphi_1}^{\varphi'=\varphi_2} \int_{\varphi''=\varphi_3}^{\varphi''=\varphi_4} f[z + iy \cos\varphi' + ix \sin\varphi' . \cos\varphi'' + iv \sin\varphi' \sin\varphi''] . \psi_1(\varphi') . \psi_2(\varphi'')\, d\varphi'\, d\varphi'' \quad (10)$$

Examples. A potential in three dimensions is

$$u_3 = \frac{1}{r_3}$$

where

$$r_3^2 = x_1^2 + x_2^2 + x_3^2 .$$

Hence a potential in four dimensions is

$$u_4 = \tfrac{1}{4} \int_0^{2\pi} \frac{d\varphi'}{\{x_4^2 + x_3^2 + (x_2^2 + x_1^2) \cos^2 \varphi'\}^{\frac{1}{2}}} = \tfrac{1}{4} \int_0^{2\pi} \frac{d\varphi'}{(r_4^2 - r_2^2 \sin^2 \varphi')^{\frac{1}{2}}} = \frac{1}{r_4} K(k),$$

where K is the complete elliptic integral with modulus

$$k = \frac{r_2}{r_4} = \left(\frac{x_1^2 + x_2^2}{x_1^2 + x_2^2 + x_3^2 + x_4^2}\right)^{\frac{1}{2}} .$$

Again, with the above notation a four dimensional potential is

$$u_4 = \frac{1}{r_4^2} ,$$

hence a five dimensional one is

$$u_5 = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{d\varphi'}{x_5^2 + x_4^2 + x_3^2 + (x_2^2 + x_1^2)\cos^2\varphi'} = \frac{1}{r_3 r_5}.$$

Let us now turn to the periodic *wave* equation in three dimensions

$$(\Delta + k^2)\, w = 0. \tag{11}$$

The method used above gives us, starting with the one dimensional solution

$$w_1 = e^{ikx}, \tag{12}$$

a solution in two dimensions.

For, (12) can be written as

$$w_1 = e^{ikx} = e^{ik\rho\cos\varphi}.$$

Hence a solution will be

$$w_2 = e^{ik\rho\cos(\varphi-\varphi')} = e^{ik(\rho\cos\varphi\cos\varphi' + \rho\sin\varphi\sin\varphi')} = e^{ik(x\cos\varphi' + y\sin\varphi')},$$

and thus a more general two-dimensional wave-solution is

$$w_2 = \int\limits_{\varphi'=\varphi_1}^{\varphi'=\varphi_2} e^{ik(x\cos\varphi' + y\sin\varphi')}.\ \psi(\varphi')\ .\ d\varphi', \tag{13}$$

where again φ_1 and φ_2 are arbitrary angles (real or complex) and $\psi\ (\varphi')$ is any arbitrary „weight" function.

Performing the same procedure a second time gives us the three dimensional wave solution

$$w_3 = \int\limits_{\varphi'=\varphi_1}^{\varphi'=\varphi_2}\int\limits_{\varphi''=\varphi_3}^{\varphi''=\varphi_4} e^{ik(x\sin\varphi''\cos\varphi' + y\sin\varphi''\sin\varphi' + z\cos\varphi'')}.\psi_1(\varphi').\psi_2(\varphi'')d\varphi'.d\varphi''. \tag{14}$$

Obviously our procedure can be extended to any number of dimensions, as was shown above for the potential case.

(14) is identical with Professor Whittaker's wave solution[3]), only the integration limits in (14) are here arbitrary, instead of 0, π, 0, 2π.

Hence (13) and (14) cover e.g. not only the ordinary Bessel-functions but also the Hankel-functions, as follows from the example below.

Examples.

a) A one-dimensional wave solution is

$$e^{ikx} = e^{ik\rho \cos \varphi}.$$

Therefore a two dimensional solution is

$$e^{ik\rho \cos (\varphi + \varphi')}$$

and hence, introducing a weight function

$$e^{in\left(\varphi' - \frac{\pi}{2}\right)},$$

another two-dimensional solution is

$$w_2 = \frac{1}{\pi} \int\limits_{-\eta + i\infty}^{\eta - i\infty} e^{ik\rho \cos (\varphi + \varphi') + in\left(\varphi' - \frac{\pi}{2}\right)} d\varphi'$$

$$= \frac{1}{\pi} \int\limits_{i\infty + \varphi - \eta}^{-i\infty + \eta + \varphi} e^{ik\rho \cos \psi + in\left(\psi - \varphi - \frac{\pi}{2}\right)} d\psi$$

$$= e^{-in\varphi} \frac{1}{\pi} \int\limits_{i\infty + \varphi - \eta}^{-i\infty + \eta + \varphi} e^{ik\rho \cos \psi + in\left(\psi - \frac{\pi}{2}\right)} d\psi$$

$$= e^{-in\varphi} H_n^{(1)}(k\rho),$$

(according to Sommerfeld's definition of the Hankel-functions [4]); and which expression in fact satisfies the two dimension al wave equation. The arbitrary angle η must be chosen such that the integral converges.

b) Returning finally to real limits and starting again with the plane (one-dimensional) wave e^{ikx}, we can at once obtain from it the two dimensional wave

$$u_2 = \int\limits_0^{2\pi} e^{ik\rho \cos (\varphi + \varphi')} d\varphi' = 2\pi J_o(k\rho).$$

Again if we superpose the plane wave homogeneously over all real directions of the solid angle 4π in three dimensions we obtain the three dimensional wave

$$u_3 = \int\limits_0^{\pi} \int\limits_0^{2\pi} e^{ikx} \sin \theta \, d\theta \, d\varphi = \int e^{ikx} d\omega_3 = (2\pi)^{3/2} \frac{J_{\frac{1}{2}}(kr)}{(kr)^{\frac{1}{2}}} = 4\pi \frac{\sin kr}{kr},$$

where $d\omega_3$ means the differential of the three dimensional solid angle.

In conclusion, integration of the plane wave over the solid angle in n dimensions yields the n-dimensional solution

$$u_n = \int e^{ikx}\, d\omega_n = (2\pi)^{\frac{n}{2}} \frac{J_{\frac{n}{2}-1}(kr_n)}{(kr_n)^{\frac{n}{2}-1}},$$

which expression, for $kr_n = 0$, reduces to

$$\frac{2\pi^{\frac{n}{2}}}{\Pi\left(\frac{n}{2}-1\right)},$$

the solid angle in n dimensions, as it ought to be.

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REFERENCES

1) Hadamard, Le problème de Cauchy, Paris 1932, p. 69.
2) Math. Ann. **57**, 333, 1903.
3) l. c.
4) See e. g. Jahnke-Emden, Funktionentafeln.